

RANK 3 ARITHMETICALLY COHEN-MACAULAY BUNDLES OVER HYPERSURFACES

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ABSTRACT. Let X be a smooth projective hypersurface of dimension ≥ 5 and let E be an arithmetically Cohen-Macaulay bundle on X of any rank. We prove that E splits as a direct sum of line bundles if and only if $H_*^i(X, \wedge^2 E) = 0$ for $i = 1, 2, 3, 4$. As a corollary this result proves a conjecture of Buchweitz, Greuel and Schreyer for the case of rank 3 arithmetically Cohen-Macaulay bundles.

1. INTRODUCTION

We work over an algebraically closed field of characteristic 0. Let $\{X, \mathcal{O}_X(1)\} \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree d . We say a vector bundle on X is *split* if it can be written as a direct sum of line bundles. We say that it is *indecomposable* if it can not be written as a direct sum of vector bundles of strictly smaller rank.

An *arithmetically Cohen-Macaulay (ACM)* vector bundle E on X is a locally free sheaf satisfying

$$H_*^i(X, E) := \bigoplus_{k \in \mathbb{Z}} H^i(X, E(k)) = 0 \quad \text{for } i = 1, \dots, n-1$$

Some of the reasons why the study of ACM bundles is important are:

- On projective space, ACM bundles are precisely the bundles which are direct sum of line bundles [Horrocks1964].
- By semicontinuity, ACM bundles form an open set in any flat family of vector bundles over X .
- The n 'th syzygy of a resolution of any vector bundle on X by split bundles, is an arithmetically Cohen-Macaulay bundle [Eisenbud1981].
- These sheaves correspond to maximal Cohen-Macaulay modules over the associated coordinate ring [Beauville2000].

When $d > 1$ there always exist indecomposable arithmetically Cohen-Macaulay bundles see e.g. [KRR2007] for low dimensional construction and [BGS1987] for a construction for higher dimensional hypersurfaces. The following conjecture forms the basis of research done in the direction of investigating the splitting behaviour of ACM bundles over hypersurfaces:

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Conjecture (Buchweitz, Greuel and Schreyer [BGS1987]): Let $X \subset \mathbb{P}^n$ be a hypersurface. Let E be an ACM bundle on X . If $\text{rank } E < 2^e$, where $e = \left\lfloor \frac{n-2}{2} \right\rfloor$, then E splits. (Here $[q]$ denotes the largest integer $\leq q$.) \square

This conjecture can not be strengthened further as the authors constructed an indecomposable ACM bundle of rank 2^e in *op. cit.*

For rank 2 ACM bundles, the conjecture follows from [Kleppe1978]. Generic behaviour for rank 2 case is also well understood when $n \geq 4$ and we refer the reader to [CM2002], [CM2004], [CM2005], [KRR2007], [KRR2007(2)], [Ravindra2009] and to the reference cited in these articles. For lower dimensional case, we refer the reader to [Madonna1998], [Madonna2000], [Faenzi2008], [CF2009] and [CH2011]. The result for rank 2 bundles was generalized to complete intersections in [BR2010].

For rank 3 ACM bundles the conjecture predicts splitting for $n \geq 5$ dimensional hypersurfaces. We proved a weaker version in [Tripathi2015]. In this article, we prove the conjecture for rank 3 arithmetically Cohen-Macaulay bundles.

Theorem 1.1. *Let X be a smooth hypersurface of dimension ≥ 5 . Let E be a rank 3 arithmetically Cohen-Macaulay bundle over X . Then E is a split bundle.*

This result follows as a corollary from the main result of this article - a splitting criterion for ACM bundles of any rank.

Theorem 1.2. *Let X be a smooth hypersurface of dimension ≥ 5 . Let E be an arithmetically Cohen-Macaulay vector bundle on X of any rank. Then E splits if and only if $H_*^i(X, \wedge^2 E) = 0$ for $i = 1, 2, 3, 4$.*

2. PRELIMINARIES

In this section, we will recall some standard facts about arithmetically Cohen-Macaulay bundles over hypersurfaces.

Let $X \subset \mathbb{P}^{n+1}$ be a degree d smooth hypersurface given by homogeneous polynomial $f = 0$. Let E be an ACM bundle of rank r on X . By Serre's duality, E^\vee is also ACM.

For notational ease, we will use \sim to denote a vector bundle on \mathbb{P}^{n+1} . By Hilbert's syzygy theorem, being a coherent sheaf on \mathbb{P}^{n+1} , E admits a finite length minimal free resolution

$$0 \rightarrow \widetilde{F}_t \rightarrow \widetilde{F}_{t-1} \rightarrow \dots \rightarrow \widetilde{F}_1 \rightarrow \widetilde{F}_0 \rightarrow E \rightarrow 0$$

where \widetilde{F}_i are direct sums of the form $\oplus_j \mathcal{O}_{\mathbb{P}^{n+1}}(a_j)$. By minimality of the resolution and the ACM condition on E , the first syzygy $\widetilde{K} = \text{Ker}(\widetilde{F}_0 \rightarrow E)$ is an ACM bundle on \mathbb{P}^{n+1} and therefore is a split bundle by Horrock's criterion. Thus the minimal free resolution of E on \mathbb{P}^{n+1} is of the form

$$0 \rightarrow \widetilde{F}_1 \xrightarrow{\phi} \widetilde{F}_0 \rightarrow E \rightarrow 0 \tag{1}$$

Localizing at the generic point, one checks that the ranks of \widetilde{F}_1 and \widetilde{F}_0 are same. Restricting the above resolution to X gives,

$$0 \rightarrow \text{Tor}_{\mathbb{P}^{n+1}}^1(E, \mathcal{O}_X) \rightarrow \bar{F}_1 \rightarrow \bar{F}_0 \rightarrow E \rightarrow 0$$

where one computes the Tor term by tensoring $0 \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \xrightarrow{\times f} \mathcal{O}_{\mathbb{P}^{n+1}} \rightarrow \mathcal{O}_X \rightarrow 0$ with E to get $\text{Tor}_{\mathbb{P}^{n+1}}^1(E, \mathcal{O}_X) = E(-d)$ as multiplication by f vanishes on X . Thus the above four term sequence breaks up as

$$0 \rightarrow E^\sigma \rightarrow \bar{F}_0 \rightarrow E \rightarrow 0 \quad (2)$$

$$0 \rightarrow E(-d) \rightarrow \bar{F}_1 \rightarrow E^\sigma \rightarrow 0 \quad (3)$$

where $\bar{F}_i = \widetilde{F}_i \otimes \mathcal{O}_X$ are split bundles over X of rank m and $E^\sigma := \text{Ker}(\bar{F}_0 \twoheadrightarrow E)$ is an arithmetically Cohen-Macaulay bundle on X .

We state the following facts (without proof) about matrix factorization theory of Eisenbud and the connection between E and E^σ . We choose a matrix (with homogeneous polynomial entries) to represent the map $\phi : \widetilde{F}_1 \rightarrow \widetilde{F}_0$ and henceforth we will use the symbol ϕ interchangeably to represent either the matrix or the map. Then

- (1) There exists an injective map $\psi : \widetilde{F}_0(-d) \rightarrow \widetilde{F}_1$ such that $\phi\psi = \psi\phi = f1$ where 1 denotes the identity matrix.
- (2) $\text{Coker}(\psi) = E^\sigma$ and E is indecomposable if and only if E^σ is indecomposable.
- (3) $0 \rightarrow \widetilde{F}_0(-d) \rightarrow \widetilde{F}_1 \rightarrow E^\sigma \rightarrow 0$ is a minimal free resolution of E^σ .

For details, we refer to section 6 of [Eisenbud1981] and section 2 of [CH2011].

Lemma 2.1. *Let f be any homogeneous (perhaps reducible) polynomial of degree d . Let $X = V(f) \subset \mathbb{P}^{n+1}$ be the vanishing set. Suppose \mathcal{F} be any coherent sheaf on X which admits a free resolution on \mathbb{P}^{n+1} of the form*

$$0 \rightarrow \widetilde{F}_1 \rightarrow \widetilde{F}_0 \rightarrow \mathcal{F} \rightarrow 0$$

where \widetilde{F}_i are direct sum of line bundles on \mathbb{P}^{n+1} . Then \mathcal{F} is a reflexive sheaf on X .

Proof. We apply $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^{n+1}})$ on the resolution of \mathcal{F} to get

$$0 \rightarrow \mathcal{H}om(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \rightarrow \widetilde{F}_0^\vee \rightarrow \widetilde{F}_1^\vee \rightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \rightarrow 0$$

First term vanishes. To compute the $\mathcal{E}xt$ term, we apply $\mathcal{H}om(\mathcal{F}, -)$ on

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}} \rightarrow \mathcal{O}_X \rightarrow 0$$

to get

$$0 \rightarrow \mathcal{H}om(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \rightarrow \mathcal{H}om(\mathcal{F}, \mathcal{O}_X) \rightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}})(-d) \xrightarrow{\times f}$$

Here the first term vanishes as before and the last map (multiplication by f) vanishes as the sheaves are supported on X . Thus we get $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \cong \mathcal{F}^\vee(d)$ and a resolution

of \mathcal{F}^\vee on \mathbb{P}^{n+1} as

$$0 \rightarrow \widetilde{F}_0^\vee(-d) \rightarrow \widetilde{F}_1^\vee(-d) \rightarrow \mathcal{F}^\vee \rightarrow 0 \quad (4)$$

Applying the whole process once again to the above resolution of \mathcal{F}^\vee we get the following resolution of $\mathcal{F}^{\vee\vee}$

$$0 \rightarrow \widetilde{F}_1 \rightarrow \widetilde{F}_0 \rightarrow \mathcal{F}^{\vee\vee} \rightarrow 0$$

Comparing with the resolution of \mathcal{F} , one gets the claim. \square

Given a short exact sequence of vector bundles $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ on a variety X , there exists a resolution of the k 'th exterior power $\wedge^k E_3$,

$$0 \rightarrow S^k E_1 \rightarrow S^{k-1} E_1 \otimes \wedge^1 E_2 \rightarrow \dots \wedge^k E_2 \rightarrow \wedge^k E_3 \rightarrow 0 \quad (5)$$

Dually, we also have a resolution of k 'th symmetric power,

$$0 \rightarrow \wedge^k E_1 \rightarrow \wedge^k E_2 \rightarrow \wedge^{k-1} E_2 \otimes S^1 E_3 \rightarrow \dots \wedge^1 E_2 \otimes S^{k-1} E_3 \rightarrow S^k E_3 \rightarrow 0 \quad (6)$$

For details we refer the reader to [BE1975].

3. A COKERNEL SHEAF

Suppose $\text{rank } \widetilde{F}_0 = \text{rank } \widetilde{F}_1 = m$. Fix any integer $k \leq \min\{\text{rank}(E), \text{rank}(E^\sigma)\}$. Let $X_k = V(f^k)$ denote the scheme-theoretic k 'th thickening of $X \subset \mathbb{P}^{n+1}$.

We consider the k 'th exterior power of the map $\phi : \widetilde{F}_1 \rightarrow \widetilde{F}_0$ in equation (1) and denote the cokernel sheaf by \mathcal{F}_k

$$0 \rightarrow \wedge^k \widetilde{F}_1 \xrightarrow{\wedge^k \phi} \wedge^k \widetilde{F}_0 \rightarrow \mathcal{F}_k \rightarrow 0 \quad (7)$$

The following lemma states some properties of the sheaf \mathcal{F}_k . Our proof is similar to that in section 2 of [KRR2007] where the case when E is a rank 2 ACM bundle and $k = 2$ was studied.

Lemma 3.1. (1) \mathcal{F}_k is a coherent \mathcal{O}_{X_k} -module where X_k is the thickened hypersurface defined scheme theoretically by f^k .

(2) $\bar{\mathcal{F}}_k := \mathcal{F}_k \otimes \mathcal{O}_X$ is a vector bundle on X of rank $\binom{m}{k} - \binom{m-r}{k}$

(3) \mathcal{F}_k is an ACM and reflexive sheaf on X_k .

Proof. First two claims can be verified locally. By localising on X , one can assume that equation (1) looks like

$$0 \rightarrow \mathcal{O}_p^{\oplus m} \xrightarrow{\phi} \mathcal{O}_p^{\oplus m} \rightarrow E_p \rightarrow 0$$

and the matrix ϕ is given by the $m \times m$ diagonal matrix

$$\{f, \dots, f, 1, \dots, 1\}$$

where f appears $r = \text{rank}(E)$ times and 1 appears $m - r$ times on the diagonal. Then locally the matrix $\wedge^k \phi$ is the diagonal matrix

$$\{f^k, \dots, f^k, f^{k-1} \dots f^{k-1}, f^{k-2}, \dots, f, 1, 1, \dots, 1\}$$

where f^{k-i} appears $\binom{r}{k-i} \binom{m-r}{i}$ times on the diagonal. In particular, locally \mathcal{F}_k is of the form

$$\mathcal{O}_{X_k}^{\oplus \binom{r}{k}} \oplus \mathcal{O}_{X_{k-1}}^{\oplus \binom{r}{k-1} \cdot \binom{m-r}{1}} \oplus \dots \oplus \mathcal{O}_{X_{k-i}}^{\oplus \binom{r}{k-i} \cdot \binom{m-r}{i}} \dots \oplus \mathcal{O}_X^{\oplus \binom{r}{1} \cdot \binom{m-r}{k-1}}$$

This proves the first claim and also that $\bar{\mathcal{F}}_k = \mathcal{F}_k \otimes \mathcal{O}_X$ is a vector bundle on X . Claim about the rank is verified by the above local description of \mathcal{F}_k and the combinatorial identity

$$\binom{m}{k} = \sum_i \binom{r}{i} \binom{m-r}{k-i}$$

By equation (7), one easily sees that \mathcal{F}_k is an ACM sheaf on X_k . Lemma 2.1 completes the proof by showing that \mathcal{F}_k is a reflexive sheaf. \square

We now restrict sequence (7) to X

$$0 \rightarrow \text{Tor}_{\mathbb{P}^{n+1}}^1(\mathcal{F}_k, \mathcal{O}_X) \rightarrow \wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0 \rightarrow \bar{\mathcal{F}}_k \rightarrow 0 \quad (8)$$

This is a sequence of vector bundles and the Tor term is a vector bundle of same rank as $\bar{\mathcal{F}}_k$. In fact, the map $F_1 \rightarrow F_0$ factors via E^σ , therefore by functoriality of exterior product, the map $\wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0$ factors via $\wedge^k E^\sigma$ and the sequence (8) breaks up as

$$0 \rightarrow \text{Tor}_{\mathbb{P}^{n+1}}^1(\mathcal{F}_k, \mathcal{O}_X) \rightarrow \wedge^k \bar{F}_1 \rightarrow \wedge^k E^\sigma \rightarrow 0 \quad (9)$$

and

$$0 \rightarrow \wedge^k E^\sigma \rightarrow \wedge^k \bar{F}_0 \rightarrow \bar{\mathcal{F}}_k \rightarrow 0 \quad (10)$$

Thus the Tor term appears as the first term in the filtration of k 'th exterior power of \bar{F}_1 derived from the sequence $0 \rightarrow E(-d) \rightarrow \bar{F}_1 \rightarrow E^\sigma \rightarrow 0$. We can say more,

Lemma 3.2. $\text{Tor}_{\mathbb{P}^{n+1}}^1(\mathcal{F}_k, \mathcal{O}_X) \cong \overline{\mathcal{F}}_k^{\vee}(-kd)$

Proof. We consider the k 'th exterior power of the minimal resolution of E^\vee given by sequence (4)

$$0 \rightarrow (\wedge^k \widetilde{F}_0^\vee)(-kd) \rightarrow (\wedge^k \widetilde{F}_1^\vee)(-kd) \rightarrow \mathcal{F}'_k \rightarrow 0 \quad (11)$$

where \mathcal{F}'_k is defined by the sequence. Restricting to X gives

$$0 \rightarrow \text{Tor}_{\mathbb{P}^{n+1}}^1(\mathcal{F}'_k, \mathcal{O}_X) \rightarrow (\wedge^k \bar{F}_0^\vee)(-kd) \rightarrow (\wedge^k \bar{F}_1^\vee)(-kd) \rightarrow \bar{\mathcal{F}}'_k \rightarrow 0$$

As in lemma 3.1 one can verify (by looking at the exterior power matrix locally) that $\bar{\mathcal{F}}'_k$ is a vector bundle and thus above is an exact sequence of vector bundles. So we can

dualize (and then twist by $-kd$) to get:

$$0 \rightarrow \bar{\mathcal{F}}'_k{}^\vee(-kd) \rightarrow \wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0 \rightarrow \text{Tor}^1(\mathcal{O}_X, \mathcal{F}'_k)^\vee(-kd) \rightarrow 0 \quad (12)$$

Comparing with equation (8), we get

$$\text{Tor}^1(\mathcal{F}_k, \mathcal{O}_X) \cong \bar{\mathcal{F}}'_k{}^\vee(-kd) \quad (13)$$

We complete the proof by showing that $\mathcal{F}'_k \cong \mathcal{F}_k^\vee$. Applying $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^{n+1}})$ to sequence (11) and simplifying as in the proof of Lemma 2.1, we get

$$0 \rightarrow \wedge^k \widetilde{F}_1 \rightarrow \wedge^k \widetilde{F}_0 \rightarrow \mathcal{F}'_k{}^\vee \rightarrow 0 \quad (14)$$

Comparing this with the sequence (7) and using the fact that by Lemma 2.1, $\mathcal{F}_k, \mathcal{F}'_k$ are both reflexive sheaves, we get that $\mathcal{F}_k^\vee \cong \mathcal{F}'_k$. \square

Lemma 3.3. *There exists a short exact sequence*

$$0 \rightarrow \wedge^k E(-kd) \rightarrow \text{Tor}_{\mathbb{P}}^1(\mathcal{F}_k, \mathcal{O}_X) \rightarrow \text{Tor}_{X_k}^1(\mathcal{F}_k, \mathcal{O}_X) \rightarrow 0$$

Proof. We restrict the sequence (7) to X_k to get a free \mathcal{O}_{X_k} -resolution of \mathcal{F}_k

$$\cdots \rightarrow \wedge^k F_1(-kd) \rightarrow \wedge^k F_0(-kd) \rightarrow \wedge^k F_1 \rightarrow \wedge^k F_0 \rightarrow \mathcal{F}_k \rightarrow 0$$

Tensoring this resolution with \mathcal{O}_X gives a complex from which we get

$$\text{Tor}_{X_k}^1(\mathcal{F}_k, \mathcal{O}_X) \cong \frac{\text{Ker}(\wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0)}{\text{Im}(\wedge^k \bar{F}_0(-kd) \rightarrow \wedge^k \bar{F}_1)} \quad (15)$$

To compute $\text{Ker}(\wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0)$, we tensor the sequence (7) with \mathcal{O}_X to get

$$\text{Ker}(\wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0) \cong \text{Tor}_{\mathbb{P}}^1(\mathcal{F}_k, \mathcal{O}_X)$$

For the $\text{Im}(\wedge^k \bar{F}_0(-kd) \rightarrow \wedge^k \bar{F}_1)$ term, we note that the map $\bar{F}_0(-d) \rightarrow \bar{F}_1$ factors via $E(-d)$ so by functoriality of wedge power,

$$\text{Im}(\wedge^k \bar{F}_0(-kd) \rightarrow \wedge^k \bar{F}_1) \cong \wedge^k E(-kd)$$

This completes the proof of the lemma. \square

3.1. A short exact sequence. Let \mathcal{F} be any coherent \mathcal{O}_{X_k} -module. The inclusions $X_{k-1} \hookrightarrow \mathbb{P}^{n+1}$ and $X \hookrightarrow X_k$ induces following short exact sequences

$$0 \rightarrow \mathcal{O}_{X_{k-1}}(-d) \rightarrow \mathcal{O}_{X_k} \rightarrow \mathcal{O}_X \rightarrow 0 \quad (16)$$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-(k-1)d) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_{X_{k-1}} \rightarrow 0 \quad (17)$$

Tensoring both sequences with $\otimes_{\mathbb{P}} \mathcal{F}$, we get

$$0 \rightarrow \text{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_{X_{k-1}}(-d)) \rightarrow \mathcal{F}(-kd) \rightarrow \text{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_X) \rightarrow \mathcal{F}|_{X_{k-1}}(-d) \rightarrow \mathcal{F} \rightarrow \overline{\mathcal{F}} \rightarrow 0 \quad (18)$$

$$0 \rightarrow \text{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_{X_{k-1}}) \rightarrow \mathcal{F}(-(k-1)d) \rightarrow \mathcal{F} \rightarrow \mathcal{F}|_{X_{k-1}} \rightarrow 0 \quad (19)$$

Similarly, tensoring sequence (16) with $\otimes_{X_k} \mathcal{F}$, we get

$$0 \rightarrow \text{Tor}_{X_k}^1(\mathcal{F}, \mathcal{O}_X) \rightarrow \mathcal{F}|_{X_{k-1}}(-d) \rightarrow \mathcal{F} \rightarrow \overline{\mathcal{F}} \rightarrow 0 \quad (20)$$

Comparing sequences (18) and (20) gives

$$0 \rightarrow \text{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_{X_{k-1}})(-d) \rightarrow \mathcal{F}(-kd) \rightarrow \text{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_X) \rightarrow \text{Tor}_{X_k}^1(\mathcal{F}, \mathcal{O}_X) \rightarrow 0 \quad (21)$$

Lemma 3.4. *With notations as above,*

$$\text{Ker}[\text{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_X) \rightarrow \text{Tor}_{X_k}^1(\mathcal{F}, \mathcal{O}_X)] \cong \text{Ker}[\mathcal{F}(-d) \rightarrow \mathcal{F}|_{X_{k-1}}(-d)]$$

Proof. Twist the sequence (19) by $-d$ and compare it with the sequence (21). \square

Proposition 3.5. *There exists a short exact sequence*

$$0 \rightarrow \wedge^k E(-(k-1)d) \rightarrow \mathcal{F}_k \rightarrow \mathcal{F}_k|_{X_{k-1}} \rightarrow 0$$

Proof. Follows from Lemma 3.3 and by putting $\mathcal{F} = \mathcal{F}_k$ in Lemma 3.4. \square

4. PROOF OF THE THEOREM

We now apply above results for $k = 2$.

Proposition 4.1. *Let E be an ACM bundle on a smooth hypersurface of dimension ≥ 3 . Then $\wedge^2 E$ is ACM if and only if $\wedge^2 E^\sigma$ is ACM.*

Proof. Assume that $\wedge^2 E$ is ACM. For $k = 2$, we get following short exact sequences for E (sequence (10) and the sequence from Lemma 3.5)

$$0 \rightarrow \wedge^2 E^\sigma \rightarrow \wedge^2 \bar{F}_0 \rightarrow \bar{\mathcal{F}}_2 \rightarrow 0 \quad (22)$$

$$0 \rightarrow \wedge^2 E(-d) \rightarrow \mathcal{F}_2 \rightarrow \bar{\mathcal{F}}_2 \rightarrow 0 \quad (23)$$

Comparing sequences (22), (23) and using the fact that $\wedge^2 \bar{F}_0, \mathcal{F}_2$ are all ACM, we get $H_*^i(\wedge^2 E^\sigma) = 0$ when $i = 2, \dots, n-1$ where $n = \dim(X)$.

To prove the vanishing for $i = 1$, we note that E^\vee is also ACM and $E^{\vee\sigma} \cong E^{\sigma\vee}(-d)$, e.g. by lemma 2.5 of [CH2011]. Therefore the same proof shows that $H_*^i(\wedge^2 (E^{\sigma\vee})) = 0$ when $i = 2, \dots, n-1$. Applying Serre's duality completes the proof. \square

We now prove our main result,

Proof of Theorem 1.2. Suffices to show one direction. Assume $H_*^i(X, \wedge^2 E) = 0$ for $i = 1, 2, 3, 4$. Consider the composition of sequences (5) and (6):

$$0 \rightarrow \wedge^2 E(-2d) \rightarrow \wedge^2 \bar{F}_1 \rightarrow \bar{F}_1 \otimes E^\sigma \rightarrow E^\sigma \otimes \bar{F}_0 \rightarrow \wedge^2 \bar{F}_0 \rightarrow \wedge^2 E \rightarrow 0$$

One concludes that $H^i(X, \wedge^2 E(k)) = H^{i+4}(X, \wedge^2 E(k-2d))$ for $i = 1, \dots, n-5$. Thus $\wedge^2 E$ is ACM. By Lemma 4.1, $\wedge^2 E^\sigma$ is also ACM. We consider sequence (5)

$$0 \rightarrow S^2 E(-d) \rightarrow E(-d) \otimes \bar{F}_1 \rightarrow \wedge^2 \bar{F}_1 \rightarrow \wedge^2 E^\sigma \rightarrow 0$$

This gives $H_*^i(S^2E) = 0$ when $i = 3, \dots, n-1$. Since $\wedge^2 E$ is ACM implies $\wedge^2 E^\vee$ is also ACM, we do a dual analysis to get $H_*^i(S^2E^\vee) = 0$ when $i = 3, \dots, n-1$. Applying Serre's duality and combining this with the vanishing for S^2E , we get that when $n-3 \geq 2$ then S^2E is also ACM.

Thus when $\dim(X) \geq 5$, $E \otimes E = \wedge^2 E \oplus S^2E$ is ACM which by Theorem 5.3 implies that E is split. \square

Remark 4.2. *We note that the statement $\wedge^2 E$ is ACM implies $E \otimes E$ is ACM is tight in the dimension. For a counterexample in lower dimension, consider any rank 2 indecomposable ACM vector bundle on a hypersurface of dimension 4. Then $\wedge^2 E$ is ACM but $E \otimes E \cong E \otimes E^\vee(t)$ can not be ACM for otherwise $H_*^2(X, \mathcal{E}nd(E)) = 0$ and hence in particular, by lemma 2.2 of [KRR2007], E is split which contradicts the indecomposability of E .*

5. $E \otimes E$ IS ACM IMPLIES E IS SPLIT

Let $f \in R = k[x_0, x_1, \dots, x_{n+1}]$ be a homogeneous irreducible polynomial of positive degree. Let $S = R/(f)$ and $X = \text{Proj}(S)$ be the corresponding hypersurface.

We state the following result without proof

Lemma 5.1. *Let E be a vector bundle on X . Let $M = H_*^0(X, E)$ be corresponding graded S -module. Then E splits if M is a free S -module.*

Following result is Theorem 3.1 in [HW1994]

Theorem 5.2 (Huneke-Weigand). *Let (R, m) be an abstract hypersurface and let M, N be R -modules, at least one of which has constant rank. If $M \otimes_R N$ is a maximal Cohen-Macaulay R -module then either M or N is free.*

The corresponding version for vector bundles is of course not true as every vector bundle on a planar curve is ACM (vacuously) and there exists indecomposable vector bundles on various planar curves. Though for our need, the following corollary suffices.

Theorem 5.3 (Corollary to Theorem 5.2). *Let $X = \text{Proj}(S)$ be a hypersurface of dimension ≥ 3 . Let E be an ACM vector bundle on X . Further assume that $E \otimes E$ is ACM. Then E splits.*

Proof. We consider a minimal resolution of E on X

$$0 \rightarrow E^\sigma \rightarrow \bar{F}_0 \rightarrow E \rightarrow 0 \quad (24)$$

and

$$0 \rightarrow E(-d) \rightarrow \bar{F}_1 \rightarrow E^\sigma \rightarrow 0 \quad (25)$$

Where \bar{F}_0, \bar{F}_1 are direct sum of line bundles. Tensoring sequence (24) with E and sequence (25) with E^σ and using the fact that $E \otimes E$ is ACM, we deduce that $E \otimes E^\sigma$ is ACM. Thus there exists a short exact sequence of graded S -modules:

$$0 \rightarrow H_*^0(E^\sigma \otimes E) \rightarrow H_*^0(\bar{F}_0 \otimes E) \rightarrow H_*^0(E \otimes E) \rightarrow 0$$

Here we are using the fact that $\dim(X) \geq 3$. Sequence (24) yields the following right exact sequence

$$H_*^0(E^\sigma) \otimes H_*^0(E) \rightarrow H_*^0(\bar{F}_0) \otimes H_*^0(E) \rightarrow H_*^0(E) \otimes H_*^0(E) \rightarrow 0$$

Thus we get the following commutative diagram

$$\begin{array}{ccccccc} H_*^0(E^\sigma) \otimes H_*^0(E) & \longrightarrow & H_*^0(\bar{F}_0) \otimes H_*^0(E) & \longrightarrow & H_*^0(E) \otimes H_*^0(E) & \longrightarrow & 0 \\ \downarrow \phi_2 & & \parallel & & \downarrow \phi_1 & & \\ 0 \longrightarrow & H_*^0(E^\sigma \otimes E) & \longrightarrow & H_*^0(\bar{F}_0 \otimes E) & \longrightarrow & H_*^0(E \otimes E) & \longrightarrow 0 \end{array}$$

where the all vertical maps are naturally defined. Middle map is an equality because \bar{F}_0 is a split bundle. By Snake's lemma, ϕ_1 is a surjective map.

Similarly we get following commutative diagram from the sequence (25)

$$\begin{array}{ccccccc} H_*^0(E(-d)) \otimes H_*^0(E) & \longrightarrow & H_*^0(\bar{F}_1) \otimes H_*^0(E) & \longrightarrow & H_*^0(E^\sigma) \otimes H_*^0(E) & \longrightarrow & 0 \\ \downarrow & & \parallel & & \downarrow \phi_2 & & \\ 0 \longrightarrow & H_*^0(E(-d) \otimes E) & \longrightarrow & H_*^0(\bar{F}_1 \otimes E) & \longrightarrow & H_*^0(E^\sigma \otimes E) & \longrightarrow 0 \end{array}$$

By Snake's lemma ϕ_2 is surjective. In turn this implies that ϕ_1 is injective and hence $H_*^0(E) \otimes H_*^0(E) \rightarrow H_*^0(E \otimes E)$ is an isomorphism. Thus $H_*^0(E) \otimes H_*^0(E)$ is a maximal Cohen-Macaulay module and we can apply Theorem 5.2 to conclude that $H_*^0(E)$ is free and therefore E splits. \square

Proof of Theorem 1.1. The perfect pairing $E \times \wedge^2 E \mapsto \wedge^3 E = \mathcal{O}_X(e)$ induces an isomorphism $\wedge^2 E \cong E^\vee(e)$. By Serre's duality then $\wedge^2 E$ is ACM and hence we can apply Theorem 1.2. \square

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REFERENCES

- [Beauville2000] A. Beauville, *Determinantal hypersurfaces*, Dedicated to William Fulton on the occasion of his 60th birthday. Michigan Math. J. 48, 39–64, 2000.
- [BR2010] J. Biswas and G.V. Ravindra, *Arithmetically Cohen-Macaulay bundles on complete intersection varieties of sufficiently high multi-degree*, Mathematische Zeitschrift 265 (2010), No. 3, 493–509.
- [BE1975] David A. Buchsbaum and David Eisenbud, *Generic free resolutions and a family of generically perfect ideals*, Adv. Math. 18, (1975) 245–301.

- [BGS1987] R.-O. Buchweitz, G.-M. Greuel, and F.-O. Schreyer, *Cohen-Macaulay modules on hypersurface singularities II*, Inv. Math. 88 (1987), 165-182.
- [CF2009] L. Chiantini and D. Faenzi, *Rank 2 arithmetically Cohen-Macaulay bundles on a general quintic surface*, Math. Nachr., vol. 282, 12 (2009) 1691-1708
- [CH2011] M. Casanellas and R. Hartshorne, *ACM bundles on cubic surfaces*, J. Eur. Math. Soc. 13 (2011), 709-731.
- [CM2002] L. Chiantini and C. Madonna, *ACM bundles on a general quintic threefold*, Matematiche (Catania) 55(2000), no. 2 (2002), 239-258.
- [CM2004] L. Chiantini and C. Madonna, *A splitting criterion for rank 2 bundles on a general sextic threefold*, Internat. J. Math. 15 (2004), no. 4, 341-359.
- [CM2005] L. Chiantini and C. Madonna, *ACM bundles on a general hypersurfaces in \mathbb{P}^5 of low degree*, Collect. Math. 56 (2005), no. 1, 85-96.
- [Eisenbud1981] D. Eisenbud, *Homological algebra on a complete intersection*, Trans. of Amer. Math. Soc. Vol. 260, No. 1 (1980), 35-64.
- [Faenzi2008] D. Faenzi, *Rank 2 arithmetically Cohen-Macaulay bundles on a nonsingular cubic surface*, J. Algebra 319 (2008), 143186.
- [Horrocks1964] G. Horrocks, *Vector bundles on the punctured spectrum of a local ring*, Proc. London Math. Soc. 14 (1964), 689-713.
- [HW1994] C. Huneke, R. Wiegand, *Tensor product of modules and the rigidity of tor*, Math. Ann., 299, 449-476 (1994).
- [Kleppe1978] H. Kleppe, *Deformation of schemes defined by vanishing of pfaffians*, Jour. of algebra 53 (1978), 84-92.
- [Madonna1998] *A splitting criterion for rank 2 vector bundles on hypersurfaces in \mathbb{P}^4* , Rend. Sem. Mat. Univ. Politec. Torino 56 (1998), no. 1, 43-54.
- [Madonna2000] C. Madonna, *Rank-two vector bundles on general quartic hypersurfaces in \mathbb{P}^4* , Rev. Mat. Complut. 13 (2000), no. 2, 287-301.
- [KRR2007] N. Mohan Kumar, A.P. Rao and G.V. Ravindra, *Arithmetically Cohen-Macaulay bundles on hypersurfaces*, Commentarii Mathematici Helvetici, 82 (2007), No. 4, 829-843.
- [KRR2007(2)] N. Mohan Kumar, A.P. Rao and G.V. Ravindra, *Arithmetically Cohen-Macaulay bundles on three dimensional hypersurfaces*, Int. Math. Res. Not. IMRN (2007), No. 8, Art. ID rnm025, 11pp.
- [Ravindra2009] G.V Ravindra, *Curves on threefolds and a conjecture of Griffiths-Harris*, Math. Ann. 345 (2009), 731-748.
- [Tripathi2015] A. Tripathi, *Low rank arithmetically Cohen-Macaulay bundles on hypersurfaces of high dimension* to appear in Comm. Algebra.

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